SOME PROPERTIES OF POLYNOMIAL CURVES.

By Frank Irwin and H. N. Wright.

The first part of the present paper develops certain simple but important properties of the curve

$$y = F(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n, \tag{1}$$

a curve which, that we may have a name for it, we shall call a *polynomial curve.** The writers have been able to find but scant account of these curves in the literature. In any case they feel that the mere fact that such a book, for instance, as Loria's *Spezielle Kurven* gives no treatment of them is almost, in itself, a justification for a paper on the subject in this place.

The second part of the paper gives, partly in application of the preceding theory, graphic constructions for the imaginary roots of a cubic equation and of certain biquadratics, a treatment of the quadratic being included for the sake of completeness.

I. A line parallel to the y-axis at a distance $-a_1/na_0$ from it, that is, the line $x = -a_1/na_0$ has important properties with reference to the curve (1). $-a_1/na_0$ is the mean value of the roots of the equation $F(x) = 0$; or, in other words, the center of gravity of the n points, real and imaginary, in which the x-axis cuts the curve, lies on the line

$$x = -a_1/na_0.$$

Let us call this line the *axis* of the curve, the point where it cuts the curve its *vertex*. Then if this axis be taken as the y-axis, the equation of the curve may be written:

$$y = f(x) \equiv p_0x^n + p_2x^{n-2} + p_3x^{n-3} + \cdots + p_n, \tag{2}$$

with the term in x^{n-1} missing. We shall, unless the contrary is stated, suppose the equation of the curve to be taken in this form.†

Now the property just stated for the x-axis is true for any line.

Proposition I. *The center of gravity of the n points, real and imaginary, in which any straight line, not parallel to the axis of the curve, cuts a polynomial curve, lies on the axis, provided n is greater than 2.*

* We shall assume, once for all, that unless the contrary is stated, n is greater than 2.

† It appears at once that the curve $y = F^r(x)$, where $F^r(x)$ denotes the rth derivative, has the same axis.

For, if the equation of the line be $y = mx + b$, the abscissas of the intersections are the roots of

$$f(x) - mx - b = 0,$$

and the mean value of these roots is zero.

Thus the axis of the curve is the diameter corresponding to any system of parallel chords; or *it is the polar line, with reference to the curve, of each point on the line at infinity.**

This property of the axis is merely a special case of the much more general and decidedly striking property stated in the following proposition.

PROPOSITION II. *The center of gravity of the mn points of intersection, real and imaginary, of any algebraic curve, $G(x, y) = 0$, of order m, and the polynomial curve* (2), *lies, when it exists at all, that is when the points are all finite, on the axis of the latter curve, always provided n is greater than* 2.

The same will be true in certain other cases also of the center of gravity of the *finite* points of intersection.

The condition that none of the mn points of intersection should lie at infinity is evidently that the coefficient of y^m in $G(x, y)$ should not be zero. Supposing this to be the case, let us consider the equation, say $g(x) = 0$, for the abscissas of the intersections, obtained by substituting the value of y from (2) in $G(x, y) = 0$. The result of this substitution in any term, $cx^h y^k$, of the latter equation will be

$$cx^h(p_0{}^k x^{nk} + kp_0{}^{k-1}p_2x^{n(k-1)} \cdot x^{n-2} + \text{terms of lower degree in } x);$$

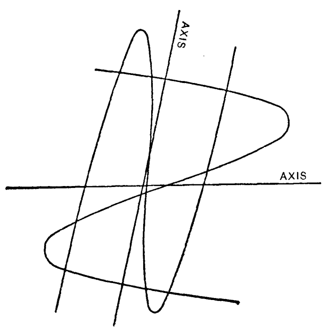
that is, the two highest terms in x will be of degrees $h + nk$ and $h + nk - 2$. In particular the term of $G(x, y)$ containing y^m, say cy^m $(c \neq 0)$, will give rise to $cp_0{}^m x^{mn} + cmp_0{}^{m-1}p_2x^{mn-2} + \cdots$; while the highest possible power of x derived from any other term of $G(x, y)$ will be the $(mn - n + 1)$th—coming from a term in xy^{m-1} (if present)—a power less than the $(mn - 1)$th, on our hypothesis that n is greater than 2. The equation $g(x) = 0$, then, will begin with $cp_0{}^m x^{mn}$, while the term in x^{mn-1} will certainly be lacking; so that the sum of the roots of this equation will be zero; which proves the proposition for this, the general case.

If, however, the term in y^m be absent from $G(x, y)$, the case will be more complicated. The degree of $g(x)$ will not now be mn; geometrically, some of the points of intersection will have gone off to infinity. The

* This suggests the question—what is the most general class of algebraic curves having such a property, the property, namely, that there exists a line l_1 each and every point on which has one and the same polar line l_2 with respect to the curve? If we project the line l_1 into the line at infinity, the line l_2 into the y-axis, it may readily be seen that the curve will reduce to the form p_0x^n + (terms of degree less than $n - 1$ in x and y) $= 0$; so that the class of curves in question are the projections of curves having equations of this type.

proposition may still hold with reference to the finite points of intersection. An example will indicate sufficiently how we should proceed.

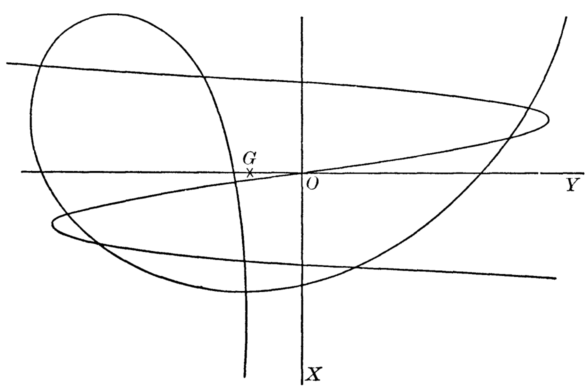
Suppose $G(x, y) \equiv ax^{10}y^2 + bx^6y^3 + cx^2y^4 + dxy^5 + e$, and let $n = 3$. *Two* terms, the first and fourth, will furnish the maximum value of $h + nk$, viz., 16. $g(x)$, then, will begin with $(ap_0^2 + dp_0^5)x^{16}$. Unless a and d are such that the coefficient of x^{16} vanishes, our proposition will not hold, for the second term of $g(x)$ will be a term in x^{15} derived from bx^6y^3; if, however, this term be not present (if $b = 0$), then the proposition holds.



A particular case in which the proposition holds is where the curve $G(x, y) = 0$ is itself a polynomial curve, if only the degree of the polynomial be less than $n - 1$. The case where it is a straight line (our first proposition) comes under this head.

Further, the center of gravity in the case of two polynomial curves whose axes are not parallel lies at the intersection of their axes (see the figure). For all their intersections are finite.

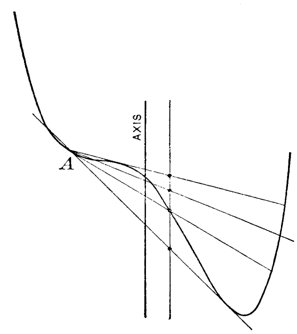
The next figure illustrates the intersections of a cubic polynomial curve, $80y = 9(3x^3 - 64x)$, with an oblique strophoid (also a cubic). It



is not a perfectly simple thing to get two such curves all of whose intersections are real. The reader may, if he wish to, verify on the drawing the fact that the sum of the abscissas of the 5 points of intersection with positive abscissas is equal to that of the 4 points with negative abscissas, and that so the center of gravity, G, actually does lie on the axis of the polynomial curve (the y-axis in this case).

Another curious property is the following:

PROPOSITION III. *If through a given point on a polynomial curve a variable straight line be drawn, the locus of the center of gravity of the* $n - 1$ *other points in which it cuts the curve is a straight line parallel to the axis of the curve* ($n > 2$).



This is evident since, calling the given point on the curve A, the center of gravity in question G, and the center of gravity of all the n points in which the straight line cuts the curve G', we have $AG/AG' = n/(n-1)$; while the locus of G' is the axis of the curve.

From this or from Proposition I follows the construction for a tangent of a cubic polynomial curve: *to draw the tangent at a point* (x_0, y_0) *of such a curve, the axis of the curve being supposed to be taken as the y-axis, join the point to that point of the curve whose abscissa is* $-2x_0$.

It also appears that *a cubic polynomial curve is symmetric with respect to its point of inflection.*

PROPOSITION IV. *A polynomial curve of order n is also of class n, and (a) the sum of the abscissas of the points of contact of the n tangents, real and imaginary, that may be drawn to it from any point, as well as (b) the sum of their slopes, is independent of the ordinate of the given point.**

Let (h, k) be the point from which the tangents are drawn, and (x_i, y_i) any one of the points of contact. The condition that the tangent at (x_i, y_i) should go through (h, k) is $k - y_i = m_i(h - x_i)$, m_i denoting the slope of the curve at the point of contact. Substituting herein the values of y_i and m_i in terms of x_i, we see that x_i is a root of

$$\begin{aligned}(n-1)p_0x^n - nhp_0x^{n-1} + (n-3)p_2x^{n-2} + \cdots \\ - 2hp_{n-2}x + k - hp_{n-1} - p_n = 0.\end{aligned} \qquad (3).$$

The sum of the abscissas of the points of tangency is the sum of the roots of (3). But this sum is seen to be independent of k, which proves (a) of the proposition. In fact, since the constant term is the only coefficient in (3) which depends upon k, we see that all of the elementary symmetric functions of the abscissas of the points of tangency, except their product, are independent of the ordinate of the point from which the tangents are drawn. To prove (b) of the proposition, since

$$m_i = np_0x_i^{n-1} + (n-2)p_2x_i^{n-3} + (n-3)p_3x_i^{n-4} + \cdots + 2p_{n-2}x_i + p_{n-1},$$

we have

* This proposition holds also when $n = 2$.

$\Sigma m_i = np_0s_{n-1} + (n-2)p_2s_{n-3} + (n-3)p_3s_{n-4} + \cdots + 2p_{n-2}s_1 + np_{n-1},$

where s_j stands for Σx_i^j, and the summation is extended to include all the points of contact. From Newton's formula,

$$q_0s_j + q_1s_{j-1} + \cdots + q_{j-1}s_1 + jq_j = 0,$$

$q_0, q_1, \cdots, q_n$ standing for the coefficients of (3), it is seen that s_j may be expressed rationally in terms of $q_0, q_1, \cdots, q_j$. Therefore Σm_i, which involves $s_1, s_2, \cdots, s_{n-1}$, but not s_n, may be expressed in terms of $q_0, q_1, \cdots, q_{n-1}$, all of which, as already noticed, are independent of k.*

For the particular case of the *quartic* polynomial curve, and for a point on the axis ($h = 0$), Prop. IV holds for the *real* tangents that may be drawn to the curve considered by themselves:† in fact, *tangents at points equidistant from the axis intersect on the axis, and the sum of their slopes is constant and equal to* $2p_3$. For, in the first place, equation (3) reduces in this case to

$$3p_0x^4 + p_2x^2 + k - p_4 = 0,$$

an equation which, if it is satisfied by x_i, is also satisfied by $-x_i$; and, secondly, the sum of the slopes of the tangents at points of the curve whose abscissas are x_i and $-x_i$ is, since

$$m_i = 4p_0x_i^3 + 2p_2x_i + p_3,$$

equal to $2p_3$. If there are points on the axis from which four real tangents can be drawn, the sum of the slopes is different for them, namely $4p_3$.

In particular, the abscissas of the two inflections, real or imaginary, being, by the second footnote to this paper, equal and of opposite sign, *the inflectional tangents meet on the axis.*

Again, the points of contact of the real double tangent, if the curve have one, are equidistant from the axis and its slope is p_3. Since this is the slope at the vertex, we have here a means of constructing, from its graph, the axis of a quartic of this kind: we find the point where the tangent is parallel to the bitangent; this is the vertex, a point on the axis.

We may include here for convenience one further property of the quartic. *The points of intersection with the curve of any line of slope* p_3, $y = p_3x + b$, *consist of pairs of points equidistant from the axis.* For the abscissas of these intersections are the roots of

* We may state a proposition still more definite than the above, which if it adds nothing very important is at all events picturesque: *as* (h, k) *moves on a line parallel to the y-axis, the center of gravity of the points of contact of the tangents from* (h, k) *moves on a parallel line with a velocity* $1/(n-1)$ *times as great.* For its ordinate, $\bar{y} = \Sigma y_i/n = (p_0s_n + p_2s_{n-2} + \cdots + np_n)/n$. Here s_n alone involves k, and $q_0s_n = -q_1s_{n-1} - \cdots - nq_n$. So that $\bar{y} = \text{constant} + p_0s_n/n = \text{constant} - q_np_0/q_0 = \text{constant} - k/(n-1)$.

† That this is not true in general may be seen easily by examining a few simple cases.

$$p_0x^4 + p_2x^2 + p_4 - b = 0,$$

which consist of pairs of numbers equal and of opposite sign. We might call the direction of these lines of slope p_3 *the direction of symmetry* of the curve.

Note. It is of interest to state the projective propositions of which ours are special metrical cases.* The polynomial curve projects into the most general curve, C, with a singular point, A, of order $n - 1$ and all the branches there tangent to one and the same straight line. On the other hand, as mentioned towards the beginning of this article, the polar line of the point at infinity on a given line cuts that line in the center of gravity of the latter's intersections with the curve. Our propositions may then be generalized as follows:

PROPOSITION I. *All the points on the tangent at A have one and the same polar line with respect to C, a line that passes through A.*

PROPOSITION II. *Join to A the intersections of C with any algebraic curve, G, of order m (that does not pass through A). Then all the points on the tangent at A have one and the same polar line with respect to this pencil of mn lines, and this polar line is independent of what curve G is chosen.*

The generalization of III offers no difficulty.

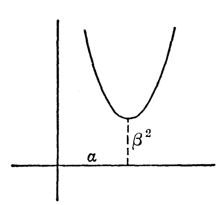
PROPOSITION IV (*a*). *If the points of contact of the n tangents from a point P to the curve C be joined to A, all points on the tangent at A have the same polar line with respect to this pencil of n lines, and the polar line does not vary so long as P stays on a given line through A.*

II. *Construction, from the graph, of the complex roots of a quadratic, cubic, biquadratic equation.*†

Quadratic. Let $\alpha + \beta i$ be the complex roots of a quadratic equation,

$$f(x) \equiv x^2 - 2\alpha x + \alpha^2 + \beta^2 = 0,$$

and suppose the graph of $y = f(x)$ constructed. Then $dy/dx = 2x - 2\alpha$, which vanishes when $x = \alpha$, while $y = \beta^2$ when $x = \alpha$. Therefore *the abscissa of the minimum point is the real part of the rootsand the ordinate is the square of the coefficient of $\pm i$.*



Cubic. Consider a cubic equation with two complex roots $\alpha \pm \beta i$,

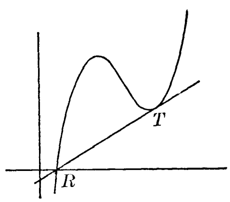
$$f(x) \equiv (x - \rho)\,(x^2 - 2\alpha x + \alpha^2 + \beta^2) = 0.$$

Let the graph of $y = f(x)$ cut the x-axis in R, then, by Prop. III, the center

* Something of the kind was suggested to us by Professor Veblen.

† See Popular Astronomy for February, 1909, where Mr. R. H. Gleason treats the quadratic and gives the construction in the text for the cubic.

of gravity of the second and third intersections of any line through R with the curve lies in a line parallel to the y-axis. Now, first, the equation of this line is $x = \alpha$, for the x-axis is a line through R cutting the cubic also where $x = \alpha \pm i\beta$, whose mean is α. And, secondly this line must pass through T, the point of contact of the tangent from R: in other words the abscissa of T is α.



Again, the slope of the curve, $x^2 - 2\alpha x + \alpha^2 + \beta^2 + 2(x - \alpha)(x - \rho)$, is β^2 when $x = \alpha$.

Hence, *the abscissa of T is the real part of the complex roots and the slope at T is the square of the coefficient of* $\pm i$.

Biquadratic. Our discussion will be confined to biquadratics with two real and two complex roots, say, $\alpha \pm \beta$ and $\gamma \pm \delta i$:

$$f(x) \equiv x^4 - 2(\alpha + \gamma)x^3 + (\alpha^2 - \beta^2 + \gamma^2 + \delta^2 + 4\alpha\gamma)x^2$$
$$- 2(\alpha\gamma^2 + \alpha\delta^2 + \alpha^2\gamma - \beta^2\gamma)x + (\alpha^2 - \beta^2)(\gamma^2 + \delta^2) = 0.$$

The axis of $y = f(x)$ is $x = (\alpha + \gamma)/2$. Then the points where $x = \alpha$, γ will, referred to the axis of the curve as y-axis, have as abscissas $(\alpha - \gamma)/2$ and $(\gamma - \alpha)/2$ respectively. Therefore, since these abscissas are equal and of opposite sign, the tangents at these points will, as proved above, meet on the axis of the curve. We find, furthermore, that when $x = \alpha$, $y = -\beta^2(\alpha - \gamma)^2 - \beta^2\delta^2$, $dy/dx = -2\beta^2(\alpha - \gamma)$. The equation of the tangent at this point of the curve is, therefore,

$$y + \beta^2(\alpha - \gamma)^2 + \beta^2\delta^2 = -2\beta^2(\alpha - \gamma)(x - \alpha);$$

and where this intersects the axis of the curve, $y = -\beta^2\delta^2$.

In constructing the roots of the equation, the axis of the curve is supposed to be known. It has been shown above how this may be constructed from the graph if the curve has a bitangent; otherwise we may construct it from the equation of the curve.

Let the curve cut the x-axis in H and K. Bisect HK at L. Then $OL = \alpha$, $LK = \beta$. Draw the tangent at A, the point of the curve whose abscissa is α, and let it intersect the axis of the curve at D. From D draw DC tangent to the curve (A and C are equidistant from the axis: this distinguishes DC from other possible tangents from D). Then the abscissa of C is γ, and the ordinate of D is $-\beta^2\delta^2$, from which δ may be found.